When The Wavepacket Is Unnecessary

L. Stodolsky
Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, 80805 München, Germany
(February 20, 1998)

We point out that many wavepacket discussions for the coherence properties of particle beams are unnecessary since they deal with stationary sources; and when the problem is stationary, essentially all information is in the energy spectrum. This recognition allows a simple answer to a number of long-debated points, usually framed in terms of "length of the wavepacket." In particular we discuss neutrino oscillations, and some issues in neutron physics. The question as to whether two simple beams with the same energy spectrum are distinguishable is answered negatively for stationary situations. The question as to whether neutrino oscillations should be thought of as taking place between states of the same energy or the same momentum is answered in favor of energy for stationary situations. Consequences for proposals involving the ^{7}Be neutrino line of the sun, the observation of oscillations in supernova neutrinos and wavepacket studies with neutrons are briefly discussed, as well as the connection with the coherence notions of quantum optics.

I. INTRODUCTION

A number of apparently subtle and difficult issues, often involving the concept of "length of the wavepacket," have long been discussed concerning the coherence properties of various particle beams.

In connection with the possibility of neutrino interference and oscillations, for example, such issues are often discussed, and there are many papers and books [1] where it is treated. In particular there has been an extensive discussion around the suggestion [2] that neutrino oscillation effects might appear in the annual variation of the earth-sun distance. For such an effect to occur, firstly, the neutrino mixing parameters must be in a favorable range. Secondly, by using an essentially monoenergetic source, the electron-capture ^{7}Be neutrino "line" from the sun, it is hoped that a washing out of the sought-for oscillations due to their energy dependence could be avoided. However, there are various line broadening and possibly other effects, and it seems the coherence properties of the neutrino flux must be understood. These have been examined in terms of the "length of the wavepacket" resulting from the electron capture process, first by Nussinov [3] and more recently re-examined [4] by him and collaborators. Other authors [5] have also looked at the point in the same way but have disagreed with some of the conclusions.

In a similar vein, we [6] tried to assess the observability of oscillation effects for the neutrinos from a supernova. This possibility arises if one envisions very small neutrino mass differences. "Normally," that is when the mass differences involved are not very small, one would expect the different mass eigenstates to separate into distinct pulses, due to their differing velocities and the great distance to the earth. Nevertheless, for extremely small mass differences the pulses could overlap upon arrival, suggesting possible oscillation effects. However, it was difficult to pursue the matter since we were uncertain what "length of the wavepacket" to use.

Again, in neutron physics, where the coherence properties of particle beams can be particularly well studied, there have been several discussions as to whether and how it could be possible to determine or observe the wavepacket properties of a beam [7,8].

In this note we would like to point out that in such problems the concern about wavepackets was actually unnecessary. For a *stationary* system all information necessary for single particle measurements is contained in the energy spectrum; and the sun or a reactor or even a supernova for most purposes, can certainly be regarded as essentially stationary sources.

This recognition also allows us to address a question which arises in these discussions: Is it possible to tell the difference between a simple beam consisting of a mixture of long wavepackets, where one might suppose a high degree of spatial coherence and an apparently more incoherent beam made out of a mixture of short wavepackets, given that both beams have the same energy spectrum? (By "simple" we mean there is no non-trivial subspace, as for mixing, see below.) This question was discussed in Refs. [7], and [8] in connection with neutron physics, and in [4] in connection with neutrinos. On the basis of various arguments and examples it was concluded that such a distinction is not possible, at least in practical experiments, although in Ref. [4] the theoretical possibility was left open.

Our conclusion will be that such a distinction is never possible, in single particle quantum mechanics and for stationary situations. This is because there is in reality no difference between the two beams.

II. STATIONARITY AND ENERGY SPECTRUM

We shall describe the beam of particles in question by means of a density matrix ρ. We make two important assumptions. First we assume that we may use a single particle description; ρ is the density matrix for a single particle. This assumption is not essential, we make it in order to simplify our discussion of stationarity and to stay in the same language as that of the discussions refered to above in neutrino and neutron physics. Thus for the moment we consider only experiments involving single particle counting, and ignore questions connected with correlations between counts, statistics effects, and multiply occupied states. However, precisely these questions are important in quantum optics, and below we examine the relation to it. Note that for real neutrino or neutron beams the density is always so low that such effects are unimportant in any case.

Secondly, and more centrally, we assume that the problem and thus the density matrix is stationary. By this we mean that no measurement on the beam described by this density matrix can show a time dependence, unless of course there are time dependent elements in the measuring device itself.

To see the implications of this assumption, we recall that the density matrix is written in terms of single particle wavefunctions ψ for the beam particle as $\rho = \sum w_i \psi_i \psi_i^*$, where w_i is the "weight" for a state i. Let us consider the time dependence of ρ. The weights w represent the properties of the source; they will be constant if we assume the source is stationary, which we do. This leaves the time development of the ψ_i as the origin of a possible time dependence, as given by the usual equation

$$\dot{\rho} = -i[H, \rho]\,, \tag{1}$$

where H is the free Hamiltonian for the beam particle.

Now, if ρ must be constant the above equation says that H and ρ commute. If ρ were not constant, then there would be in principle some measurement which would show a time variation. It seems self-evident for thermal sources like the sun or sources like reactors that no such measurement is possible. (However the notion of stationarity may not be trivial when we go beyond the single particle problem; see the remarks below on the coherent or Glauber state.)

Now since ρ commutes with H, it can be diagonalized in the energy basis. This means that up to obvious degeneracies such as direction or polarization, the beam is entirely characterized by the diagonal elements of ρ in the energy basis. But this is simply the energy spectrum. In other words, ρ is entirely determined by the energy spectrum.

We conclude that given an energy spectrum and stationary conditions, no detailed discussion of production mechanisms is necessary. Furthermore, two stationary beams with the same energy spectrum have the same density matrix and so cannot be distinguished (by single counting experiments, see below).

III. TIME AVERAGING BY THE DETECTOR

Even if the beam is time dependent, there will be many cases where this time dependence plays no role. Experiments with a chopper in a neutron beam or at a pulsed accelerator are obviously equivalent to those with a continuous beam, if time-of-flight or other timing information is not used. If the detector contains no time-dependent elements, that is performs a time average, any effects involving off-diagonal elements of ρ, $\rho_{E,E'} \sim e^{i(E-E')t}$ will be averaged to zero by the integration over time. Then, as for the stationary beam, only the diagonal elements of ρ enter into the result and all relevant information is again given by the energy spectrum.

On the other hand, if the detector has time-dependent elements, as when we use timing information, these may "beat" with $e^{i(E-E')t}$ so that off-diagonal elements indeed play a role. Implicit in these arguments is the assumption that any output or result is linear in the input density matrix, but this is a fundamental feature of quantum mechanics.

IV. DENSITY MATRIX

The bothersome feeling that more than just the spectrum ought to be involved is perhaps traceable to the somewhat unintuitive character of the density matrix in quantum mechanics. A single, given, density matrix can arise in different ways, especially when incoherence is involved. An unpolarized spin 1/2 object is equally well a mixture of spin-up and spin-down states on the one hand or a mixture of spin-left and spin-right on the other; an unpolarized photon beam is just as well a mixture of two linear or a mixture of two circular polarizations; and so on.

Similarly in the present problem: the main point is the absence of off-diagonal energy correlations in the density matrix. This might be thought of as arising in various ways; nevertheless once we know that the density matrix is stationary the results of these different ways are all equivalent. That is, given a stationary density matrix $\rho = \sum w_i \psi_i \psi_i^*$ and a stationary density matrix ρ' made up of different states and different weights $\rho' = \sum w_i' \psi_i' \psi_i'^*$ but in such a way that both have the same energy spectrum, the two are in fact equal, $\rho = \rho'$. Thus there is no way—at least in the usual understanding of quantum mechanics—to ascribe a difference between a stationary, single particle beam which is a mixture of short wavepackets on the one hand and one which is a mixture of near-plane waves on the other, if the energy spectra are the same.

Since the same density matrix may be thought of as arising in various ways, for practical considerations, we can view it in the most convenient form. This will usually be as a mixture of energy eigenstates for stationary problems, and in the following we always take an incoherent average over the energy spectrum.

V. PARTICLE MIXING

We now turn to particle mixing, as for neutrinos, K^0's or other neutral heavy flavor mesons.

Energy or momentum?—In mixing problems, where we have to deal with linear combinations of particles of different mass, the question comes up as to whether one should deal with states of the same energy or the same momentum. Since as stated above, for stationary conditions we are to perform the calculation as an incoherent sum over energies, we have given the answer "energy." Evidently, for stationary problems it is most natural to use stationary wavefunctions $\sim e^{-iEt}$.

Non-Trivial Subspace.—But here the density matrix, although diagonal in energy for stationary conditions, will have a non-trivial subspace for a given energy. That is, the density matrix element at a given energy will in general be a matrix, say a 2×2 matrix for a two state system, and in many cases this matrix will be non-trivial. Thus for mixing problems we must qualify our statement that "all information is in the energy spectrum" and consider how to determine this matrix.

The most commonly discussed situation is that of emission of a state with a definite quantum number ("flavor") but which does not necessarily correspond to a definite mass. We concentrate on this case. In contrast to the kinematic variables, we then have a pure state with respect to the internal ("flavor") variables. We may have, for instance, that the neutrino is emitted by nuclear beta-decay, as a ν_e; or for kaons with some flavor tag, as say a $\bar{K}^0$. The problem now is to determine this pure state. We do this by using that fact it is fixed at emission [9], at $z = 0$, where z is the spatial coordinate. For a two-state system as with two neutrino species or with K^0's, we have in terms of ordinary spatial wavefunctions e^{ipz} and internal spinors U:

$$\psi = \alpha U_1 e^{ip_1 z} + \beta U_2 e^{ip_2 z} \qquad (2)$$

where 1 and 2 refer to the mass eigenstates so $p_1^2 = E^2 - m_1^2$ and $p_2^2 = E^2 - m_2^2$, and the U's are the mass eigenstates. The coefficients α and β are now so chosen that $\alpha U_1 + \beta U_2$ give the desired state at emission, and so oscillations take place in space due to the non-vanishing difference between p_1 and p_2. Oscillation effects at a given detection point are then calculated as an incoherent average over E.

Decoherence.—In the foregoing case we had a pure state for each energy, given by the wavefunction Eq. (2) and so a correspondingly simple density matrix in the 2×2 subspace. Now the density matrix should in principle be calculated from the details of the various production processes. One may ask the following question [10]: Suppose it were possible to distinguish which neutrino mass eigenstate is emitted by detection of the recoils in the emission process. Or equivalently we can ask what happens if the surrounding medium reacts very differently according to which mass eigenstate is emitted. Are there then oscillation phenomena?

This is a question of "quantum damping" or "decoherence" [11]. It corresponds, in the extreme case, to conditions in which it would not be possible to form the coherent initial state Eq. (2). It results from the fact that in forming the density matrix for the beam, we are instructed to average (or "trace") over the many unobserved variables of the source or equivalently the "recoil detectors." Now if the conditions are such that the unobserved variables go into different states according to which neutrino mass eigenstate is emitted, the result will be a strongly mixed or incoherent state for the beam and not a pure state like Eq. (2). In the extreme case of strong damping and so no coherence between mass eigenstates 1 and 2, the density matrix in the mass eigenstate basis, for an emitted ν_e would have zero off-diagonal elements, reflecting no coherence between mass eigenstates. The diagonal elements would have the values $\cos^2\theta$ and $\sin^2\theta$ in terms of the usual mixing angles [12], reflecting the amount of ν_e in the two mass eigenstates. The state breaks up into an incoherent mixture of mass eigenstates, since the mass has been "measured" [11], and there will be no oscillations.

For the very small mass differences usually contemplated in mixing processes, however, the resulting differences in momentum are so small relative to the spread of momentum in the surroundings (detailed calculations are possible by the methods of Ref. [11]) that the resulting decoherence will be negligible. Note analogous results will follow if some property other than mass is "measured" by the surroundings, in which case there would be oscillations, in general.

VI. GENERAL REMARKS

With these points in mind we can now deal with some of the issues raised in the introduction.

Lines, Continua, and "Length of the Wavepacket".—From the present point of view—always assuming stationarity—a "line" is simply a strong source in a narrow energy range, and no particular coherence properties should be assigned to it. The "length of the wavepacket," if we wish to use that language, is simply determined by the energy (strictly, momentum for non-relativstic particles) band used in the data sample. For a "line" we are, aside from fine points, essentially interested in the width, regardless of how it originates.

In principle, then, one can achieve the same results as for a "line" with a broad source and energy selection by the detector, if the resolution of the detector and the number of events is adequate. For example, solar neutrino detectors with some degree of energy resolution could possibly look for oscillations in other regions of the spectrum in the same way as in the ^{7}Be proposal (see below).

Distinguishability of Beams.—We stress that in our usual understanding of quantum mechanics there is no way to even ascribe a difference to two simple stationary beams with the same energy spectrum, since they have one and the same density matrix. Hence any experiment, even one involving detectors with time dependent elements [8], which could establish such a difference would be of the utmost interest. This is of course not limited to neutron physics. An experiment which could tell, say, if an unpolarized photon beam were made of a mixture of circular and not linear polarizations would also be very surprising.

VII. NEUTRINO OSCILLATIONS

Turning now to searches for neutrino oscillations, the stationarity assumption must certainly be good for the sun or reactors. Even a supernova, evolving relatively rapidly, can be taken as a sequence of approximately thermal, quasistationary states. Thus we calculate all effects by an incoherent average over energy.

Coherence and energy resolution.—Since we need only take an incoherent average over the energy spectrum, the only relevant question for the observability of neutrino oscillations is the energy spread ΔE in a sample. That is, ΔE must be sufficiently small so that the energy dependent phase difference $\phi_1 - \phi_2$ for mass eigenstates 1 and 2 does not vary by more than 2π over the sample. At a distance d from the source and in terms of the oscillation length parameter l given by the inverse of the momentum difference for relativistic particles $1/l = \Delta M^2/2E$, one has $\phi_1 - \phi_2 = d/l$. This leads to the requirement

$$\Delta E/E < 2\pi l/d\,. \tag{3}$$

(Taking l much larger than the dimensions of the source, otherwise there is a further average over d.)

Separation of Packets.—Viewed classically, neutrino states with different masses will move apart because of their differing velocities, and in Ref. [6] the fact that for supernovas the very long flight paths can lead to a macroscopic separation of wavepackets was analyzed. In the case of a macroscopic separation there will obviously be no interference effects between mass eigenstates. On the other hand if the classical separation is very small, as for most terrestial experiments, one presumably need not discuss this problem. Still, one might wonder how small is small enough and in the case of small or partial separation, if some additional treatment is perhaps necessary.

None however is in fact needed, at least in the stationary case, since all effects are taken care of by the average over the energy spectrum. Since the "length of the wavepacket" is simply given by the band of energies in the sample, the question of the "separation" can be viewed as simply another form of the resolution condition Eq. (3): Using $v_1 - v_2 \approx 1/lE$ the classical separation s is d/lE so Eq. (3) can be written as

$$\Delta E < 2\pi/s \tag{4}$$

which is the usual result in quantum mechanics that there is no interference if the two mass packets separate by more than the inverse of the available resolution. This also indicates, as usual, that "separation of the wavepackets" can be compensated by an increase in the experimental resolution. Were Mössbauer effect-like detection and resolution ever possible for neutrinos, then very small l could be studied for various sources. Note that even a very great energy resolution does not necessarily imply the ability to distinguish mass eigenstates, unless the associated momentum difference can be manifested in some way (see the discussion on "decoherence").

The ^{7}Be line.—Here ΔE is determined by the spread of the line, which a rough thermal estimate would give as about 1 KeV. A detailed calculation by Bahcall [13] roughly verifies this, but results in a more complicated and asymmetric line shape.

In "length of the wavepacket" language the "length" here is then about 1/KeV or 2×10^{-8} cm (natural units), not too far from the "length of the wavepacket" estimate of 6×10^{-8} cm given in Ref. [3], (but somewhat different from that of Ref. [5]). Hence the practical consequences should be about the same as in Ref. [3], but for detailed analysis one can use the exact line shape. We stress that it would be wrong to consider "coherent" effects like the natural line width and "incoherent" effects like Doppler broadening on a different footing, only the full energy spectrum is of interest.

The main question here is how small a candidate l can be before the condition Eq. (3) is violated. With the figure of about 1 KeV we have $\Delta E/E \approx 1$ KeV/1 MeV $= 10^{-3}$, so Eq. (3) is easily satisfied for $l/d \approx .035$, which is the seasonal variation in the earth's orbit. To study the possibilites for smaller l with precision [14], one can integrate over the lineshape of Ref. [13].

Resolution by the Detector.—As mentioned above, we might also consider looking at other regions of the solar spectrum using the resolution of the detector. This requires at least $\Delta E/E \approx 2\pi(.035) \approx .2$, perhaps not totally impracticable, depending on the detector.

Oscillations for supernova neutrinos.—Here l must be very big to be observable, since d is so large, and to see an effect by moving the detector it must be moved a distance comparable to l. Thus a search for oscillations by moving the detector, or rather with detectors in different locations, does not come into question. However, since we now see that the "length of the wavepacket" poses no problem, and that the only condition is Eq. (3), we can return to the proposal of Ref. [6], where one uses the energy resolution of the detector to look for oscillations as function of energy. Given some degree of energy resolution, one divides the events from a supernova burst into energy bins, and is sensitive to oscillation lengths according to Eq. (3). Certainly the condition $l \sim d$ seems very improbable for an supernova; on the other hand it is amusing that there is a way, in principle, to see such tiny mass differences.

VIII. QUANTUM OPTICS

Our main point has been that for a stationary density matrix all information is in the energy spectrum. This is generally true—up to the question of non-trivial degeneracy—but it should be borne in mind that in quantum optics, there is another dimension to the energy spectrum. For particle beams like neutrons and neutrinos we usually ignore the possibility of effects related to degeneracy or fermi statistics—certainly a very good approximation—so that the energy spectrum simply refers to the distribution of single particle energies. The "energy spectrum" and the "color spectrum" are the same. On the other hand in quantum optics (where a state of three red photons can have the same energy as a state of one blue photon) there is also an energy spectrum for a given color or mode of the field, namely the distribution of photon number in the given mode [15].

The role of the single particle density matrix is played by the first order correlation function [16], G^1 which determines the intensity or single counting rate. Here again it is possible to see that if this quantity is constant in time it is determined simply by the intensity spectrum, that is by the average number of particles in each mode, paralleling the situation for single particle quantum mechanics.

When we consider the possibilty of multiply occupied states and counting correlation measurements however, there is a new aspect in that this single counting quantity may be stationary although the state as a whole is not. Consider a single mode in the Glauber or coherent state. The average number of photons or the single counting rate is constant. But the *field* has a time dependence, so in principle there is a measurement (e.g. of the electric field) showing a time dependence. Here there are significant phase relations between states with different occupation numbers; there is more information than simply the occupation numbers. (Non-stationarity should not really be surprising here since at low frequencies, that is in the maser, the coherent state is in fact used as a clock.) On the other hand the "chaotic state" of quantum optics, which resembles the thermal state, is completely characterized by its occupation numbers, and any observable is stationary, not just the single counting rate.

This point is significant in qualifying the question as to the distinguishability of two beams with the same energy spectrum. If we allow for the study of correlations between counts, then, as for example in the "bunching" of photons in quantum optics, it might be said that "packets" are observable, and thus that two beams with the same intensity spectrum could nevertheless be distinguished by such correlations. However these "bunches" are not the wavepackets of single particle quantum mechanics, applicable to low density beams, for which the discussion of distinguishability originally was intended.

ACKNOWLEDGMENTS

I am indebted to H.J. Lipkin for several discussions and help in clarifying the discussion of mixing. I would also like to thank G. Raffelt for many discussions and for stressing that the energy spectrum ought to be enough, as well as R. Glauber for a similar remark, and finally R. Golub for a discussion concerning neutron physics.

---

[1] See for example C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics* (Harwood Academic Publishers).
[2] Apparently due to Pomeranchuk; V. Gribov and B. Pontecorvo, Phys. Lett. B **28**, 493 (1969); B. Pontecorvo, Sov. Phys. JETP **26**, 984 (1968).
[3] S. Nussinov, Phys. Lett. B **63**, 201 (1976).
[4] K. Kiers, S. Nussinov and N. Weiss, Phys. Rev. D **53**, 537 (1996)
[5] L. Krauss and F. Wilzcek, Phys. Rev. Let. **55**, 122 (1985).
[6] P. Reinartz and L. Stodolsky, Z. f. Phys. C **27**, 507 (1985).
[7] H.J. Bernstein and F.E. Low, Phys. Rev. Lett. **59**, 951 (1987).
[8] R. Golub and S.K. Lamoreaux, Phys. Lett. A **162**, 122 (1992).
[9] This is the approach taken by Y. Grossman and H.J. Lipkin, Phys. Rev. D **55**, 2760 (1997), see also the following reference.
[10] H.J. Lipkin, Phys. Let. **B 348**, 604 (1995).
[11] For an introduction to these notions and methods for calculating the damping see L. Stodolsky, "Quantum Damping and Its Paradoxes", in: *Quantum Coherence*, ed. by J.S. Anandan (World Scientific, Singapore, 1990), and G. Raffelt, G. Sigl and L. Stodolsky, Phys. Rev. Lett. **70**, 2363 (1993).
[12] See G.G. Raffelt, *Stars as Laboratories for Fundamental Physics* (Univ. Chicago Press, 1996).
[13] J.N. Bahcall, Phys. Rev. **D49**, 3923 (1994).
[14] This has been done by S. Pakvasa and J. Pantaleone, Phys. Rev. Lett. **65**, 2479 (1990), using an approximate model of the asymmetric line shape.

[15] For the treatment of particle mixing in a second quantized framework with inclusion of many-particle effects see G. Raffelt, G. Sigl and L. Stodolsky Phys. Rev. **D45**,1782 (1992).

[16] R.J. Glauber, in: *Quantum Optics and Electronics, Les Houches 1964*, ed. by C. DeWitt, A. Blandin and C. Cohen Tannoudjhi (Gordon and Breach, New York).